Reply to "Comment on 'Optimal probe wave function of weak-value amplification' "

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It is pointed out that the "counter example" presented in the Comment is a family of probe wave functions which are increasingly broad as the shift becomes large. Furthermore, the author's variational calculation is not correct in the sense that we have to gauge fix the freedom of the phase translation. It is shown that there are two kinds of solutions, normalizable and un-normalizable. The former is our optimal solution, and the latter is what he found. It seems only the former is relevant from a practical point of view.

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The Comment written by Di Lorenzo [1] is nice for critically looking at the same problem from different angles. This would not only clarify the points but also would shift the emphasis to more appropriate ones than presented in our original paper [2]. His Comment is in this category, although we disagree with his conclusions. Throughout our Reply, the notation is used in Ref. [1].

We agree that the trial function (3) in Ref. [1] gives an arbitrarily large shift in the position in proportion to the parameter α. However, the variance in the position becomes much larger in proportion to the parameter α^2. Namely, the probe wave function becomes broader as its center becomes large. In our final optimal probe wave function (18) in Ref. [2], the variance becomes infinity except for $\langle \hat{x} \rangle_f = 2m$ ($m \in \mathbb{Z}$). Therefore, both cases are practically useless because of the large variance while we obtain the large shift. It is also noted that the shift on his trial function can exceed our optimal shift since our optimal solution gives the stationary value for the shift optimality.

However, in this exceptional case, $\langle \hat{x} \rangle_f = 2m$ ($m \in \mathbb{Z}$), our final optimal probe wave function with normalization is the Kronecker δ as

$$\begin{aligned} \tilde{\xi}_f(x = 2n) &= \frac{2}{\pi} \frac{\sin\left[\frac{\pi}{2}(2n - 2m)\right]}{2n - 2m} \\ &= \frac{\sin\left[(n - m)\pi\right]}{(n - m)\pi} \\ &= \delta_{mn} \quad (n \in \mathbb{Z}). \end{aligned} \tag{1}$$

Therefore, this gives zero variance. As mentioned in the main text of Ref. [2], it is remarked that the domain of our probe wave function in the momentum representation is finitely bounded

$$|k| \leq \pi/2, \tag{2}$$

with

$$|\xi_i(k = \pi/2)|^2 = |\xi_i(k = -\pi/2)|^2, \tag{3}$$

which implies the periodic boundary condition for the wave function $\xi_i(k)$ up to phase so that the position operator $\hat{x}$ has a discrete spectrum for our final optimal probe wave function.

To obtain the sharp final distribution with the large shift, the particular case, $\langle \hat{x} \rangle_f = 2m$ ($m \in \mathbb{Z}$), is important, which can be achieved by tuning the pre- and postselected states, i.e., the weak value. We believe that the probe wave function is more practical if the shift is larger while the variance is smaller, although we do not rule out the usefulness of a broad wave function of the large shift.

As he correctly remarked, the quantity $\Delta\langle \hat{x} \rangle$ is invariant under the operation $\xi_i \to e^{ix_0 k}\xi_i$ for any $x_0 \in \mathbb{R}$. Due to this gauge symmetry, the functional $\Delta\langle \hat{x} \rangle$ is flat in this phase shift direction in the function space so that it does not have a local extremum. A standard method to cope with this problem is the gauge fixing. More precisely, the difference $\langle \hat{x} \rangle_f - \langle \hat{x} \rangle_i$ is gauge invariant and coincides with $\langle \hat{x} \rangle_f$ by imposing gauge fixing condition $\langle \hat{x} \rangle_i$.

To do this, we add a Lagrange multiplier term,

$$-\,\mathrm{Im}\left[\mu \int dk\, \xi_i^* \xi_i'\right], \tag{4}$$

where μ is the Lagrange multiplier. Upon the variation with respect to μ, we obtain $\langle \hat{x} \rangle_i = 0$.

Then, Eq. (A4) in Ref. [1] should be replaced by

$$\Delta\langle \hat{x} \rangle = -\mathrm{Im}\left\{ \frac{\int dk [B\xi_i]^* [B\xi_i]'}{N_f} - \tilde{\mu} \int dk\, \xi_i^* \xi_i' \right\}, \tag{5}$$

where the combination $\tilde{\mu} := \mu + 1/N_i$ can be taken as an independent variable.

The rest of the calculation is straightforward, and Eq. (A6) in Ref. [1] is reproduced except that the quantity $|\bar{B}|^2$ is replaced by $\tilde{\mu} N_f$ and $\langle \hat{x} \rangle_i = 0$. If $\tilde{\mu} \neq 0$, the solution becomes un-normalizable as the author pointed

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out. However, if $\tilde{\mu} = 0$, our optimal solution (15) in Ref. [2] is reproduced, which is normalizable.

We think that only the normalizable and stationary solution for the probe wave function is relevant so that the case $\tilde{\mu} \neq 0$ is excluded. The point is the gauge fixing, which makes the multiplier variable.

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[1] A. Di Lorenzo, arXiv:1210.3274 to be published from Phys. Rev. A.

[2] Y. Susa, Y. Shikano, and A. Hosoya, Phys. Rev. A **85**, 052110 (2012).